Topology F'10 Name:
Midterm

1. a) Define what it means for a topological space X to be Hausdorff.

X is Hausdorff $\iff$ $\forall x, y \in X$ with $x \neq y$, $\exists$ open U_x, U_y with $x \in U_x$ & $y \in U_y$ and $U_x \cap U_y = \emptyset$

b) Give an example of a toplogical space that is not Hausdorff (you need to provide a brief explanation why it is not Hausdorff).

$X = \mathbb{R}$ with the trivial topology (the only open set containing any point is X, so its impossible to separate distinct points).

2. Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

This will be done by showing that $(A \times B)^c$ is open.

$$(A \times B)^c = \{ (a,b) : a \notin A \ \underline{\text{or}} \ b \notin B \}$$
$$= \{ (a,b) : a \notin A \} \cup \{ (a,b) : b \notin B \}$$
$$= \underbrace{(A^c \times Y)}_{\text{open}} \cup \underbrace{(X \times B^c)}_{\text{open}}$$

Since the union of open sets is open it follows that $(A \times B)^c$ is open.

3. a) Let X be a set and let $\mathbb{B}$ be a collection of subsets of X. Define what it means for $\mathbb{B}$ to be a basis.

$\mathbb{B}$ is a basis if

1) $\forall x \in X \ \exists B \in \mathbb{B} \ni x \in B$

and 2) $B_1 \cap B_2 \neq \emptyset$ and $x \in B_1 \cap B_2 \Rightarrow \exists B \in \mathbb{B} \ni x \in B \subset B_1 \cap B_2$

b) Prove the following theorem: Let X be a set and let $\mathbb{B}$ be a basis for a topology on X. In this case, U is open if and only if for each $x \in U$ there exists a basis element $B_x \in \mathbb{B}$ such that $x \in B_x \subset U$.

($\Rightarrow$) U open $\Rightarrow U = \bigcup B_\alpha$ for $B_\alpha \in \mathbb{B}$ (this is because $\mathbb{B}$ generates the topology)

So, $x \in U \Rightarrow \exists \alpha \ni x \in B_\alpha \subset U$

($\Leftarrow$) consider the union of the B_x's, i.e., let $V = \bigcup_{x \in U} B_x$. Since $B_x \subset U \ \forall$ ~~$x \in$~~ x then $V \subset U$. Also, if $V \neq U$ then $\exists x \notin V$ with $x \in U \Rightarrow \exists B_x \in \mathbb{B} \ni x \in B_x \subset U$ (but this violates our union over <u>all</u> such B_x's) $\therefore V = U$. The fact that V is open follows since its the union of open sets.

4. For a given (and fixed) number p, let T be the collection of sets that consists of $\mathbb{R}$ and all subsets of $\mathbb{R}$ that exclude p. In other words, $U \in T$ if and only if $p \notin U$ or $U = \mathbb{R}$.

a) Prove that T is a topology.

i) $\phi \overset{?}{\in} T$: yes because $p \notin \phi$

ii) $\mathbb{R} \overset{?}{\in} T$: yes, by definition

iii) $U_1, U_2 \in T \overset{?}{\Rightarrow} U_1 \cap U_2 \in T$

pf: $p \notin U_1$ & $p \notin U_2 \Rightarrow p \notin U_1 \cap U_2 \Rightarrow U_1 \cap U_2 \in T$

iv) $U_\alpha \in T \overset{?}{\Rightarrow} \cup U_\alpha \in T$

pf: if $p \in \cup U_\alpha$ then $\exists \alpha \in p \in U_\alpha$ ✱

b) Suppose A is a subset of $\mathbb{R}$ that contains p. What is $\mathrm{Int}(A)$ and what is $\mathrm{Cl}(A)$?

$$\mathrm{Int}(A) = \begin{cases} A - \{p\} & \text{if } A \neq \mathbb{R} \\ \mathbb{R} & \text{if } A = \mathbb{R} \end{cases}$$

pf: $A = \mathbb{R}$, and $\mathbb{R}$ open $\Rightarrow \mathrm{Int}(A) = \mathbb{R}$

$A \neq \mathbb{R} \Rightarrow A - \{p\}$ open $\Rightarrow A - \{p\} \subset \mathrm{Int}(A)$

given that A is not open the result follows

$\mathrm{Cl}(A) = A$: because $p \notin A^c$ then A^c is ~~closed~~ open

$\Rightarrow A$ closed $\Rightarrow \mathrm{Cl}(A) = A$

5. The following statements are false. Provide an example for each which proves that it is false (you also need to provide a brief explanation why your example works). It is assumed here that A and B are subsets of a topological space X.

a) If $x_i \in A$ and the sequence $\{x_i\}$ converges, then there is a unique point $x \in \mathrm{Cl}(A)$ such that $x_i \to x$.

$X = \mathbb{R}$ $\quad A = \mathbb{R}$ $\quad x_i = \frac{1}{i}$

$T =$ trivial top.

the only (nonempty) open set is $X \Rightarrow$ every seq converges to every point in $X \Rightarrow$ limit is not unique

b) A possible topology for X is $T = \{\phi, X, A, \partial A\}$.

$X = \mathbb{R}$ $\quad A = (0,1)$ $\quad \partial A = \{0,1\}$

$A \cup \partial A = [0,1] \notin T$

$\Rightarrow$ not topology

c) For every A and B, $\text{Int}(A \cup B) = \text{Int}(A) \cup \text{Int}(B)$

$A = \mathbb{Q}$, $B = \mathbb{I}$ $\Rightarrow$ $\text{Int}(A) = \text{Int}(B) = \emptyset$

$\text{Int}(A \cup B) = \text{Int}(\mathbb{R}) = \mathbb{R}$

d) Given any closed sets A_α, $\cup A_\alpha$ is closed.

using the st. top on $\mathbb{R}$

$A_n = [\frac{1}{n}, 1] \Rightarrow \cup A_n = \underbrace{(0,1]}_{\text{not closed}}$